# Noise-induced breakdown of coherent collective motion in swarms

Alexander S. Mikhailov
*Fritz-Haber-Institut der Max-Planck-Gesellschaft, Faradayweg 4-6, 14195 Berlin, Dahlem, Germany*

Damián H. Zanette
*Consejo Nacional de Investigaciones Científicas y Técnicas, Centro Atómico Bariloche and Instituto Balseiro, 8400 Bariloche, Argentina*
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We consider swarms formed by populations of self-propelled particles with attractive long-range interactions. These swarms represent multistable dynamical systems and can be found either in coherent traveling states or in an incoherent oscillatory state where translational motion of the entire swarm is absent. Under increasing the noise intensity, the coherent traveling state of the swarms is destroyed and an abrupt transition to the oscillatory state takes place. [S1063-651X(99)11309-6]

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There is a large class of problems where individual interacting particles, which constitute a system, are capable of active motion and form collectively traveling populations. Self-propulsion of particles is already possible in simple physical systems (see, e.g., [1–5]) and is widely found in biology where individual animals may group themselves into swarms, fish schools, bird flocks, or traveling cell populations [6–9]. The role of individual self-propelled ''particles'' can also be played by localized patterns (spots) in reaction-diffusion systems. A bifurcation leading to the onset of a translational motion of spots has been studied in an activator-inhibitor system with global feedback [10] and in three-component reaction-diffusion systems [11,12]. Interactions between individual self-propelled spots have been determined from the underlying reaction-diffusion equations and used to describe formation of bound states of such ''particles'' [13].

Mathematical modeling of collective active motion follows several different directions. One approach is based on the notion of discrete stochastic automata [8,14–16]. Another approach is formulated in terms of continuous velocity and density fields and essentially treates a swarm as an active fluid [17] (such hydrodynamical equations may be derived by averaging from the respective automata models [18]). A similar hydrodynamic approach is also used in the theory of traffic flows [19,20]. Alternatively, one can specify dynamical equations of motion for all individual particles that explicitly include interactions between them and/or action of external fields [3,7,9,13,21]. An interesting problem related to statistical mechanics of large populations of self-propelled particles is the spontaneous development of coherent collective motion in such systems. This problem has recently been discussed in the framework of continuous hydrodynamical and discrete automata models, and the properties of the respective kinetic phase transition were numerically and analytically investigated [17,18]. Both in one- and two-dimensional systems, first- and second-order transitions have been found [15].

In the present paper we consider a population of identical self-propelled particles near a transition between disordered oscillating motion and coherent translational motion. The particles interact via an isotropic attractive binary potential and are subjected to the action of noises. This globally coupled population forms a cloud (the swarm) in the considered one-dimensional space. The swarm can be found in different states. Coherent compact traveling states are characterized by a narrow distribution of velocities around a certain mean drift velocity, directed either to the left or to the right. Another possible state of this population corresponds to the absence of coherent translational motion, with noisy oscillations around a certain mean position in space, determined by the initial conditions.

The coherent traveling states exist only for sufficiently weak noise and, as the noise intensity increases, the swarm undergoes a transition to the incoherent oscillatory state. We find that the breakdown of coherent collective motion in this system is abrupt and characterized by a strong hysteresis. Thus, the globally coupled swarm represents a multistable system that may be found in different states depending on the initial conditions. This behavior, revealed by numerical simulations, is well reproduced by an approximate analytical theory and may represent a typical property of swarms with long-range interactions.

To formulate the model, we note that if a system is close to the onset of active motion and this instability is soft, i.e., characterized by a supercritical bifurcation, the motion with small velocity $V$ can generally be described by equation

$$\dot{V}=\alpha V-\beta V^3, \tag{1}$$

with real coefficients $\alpha$ and $\beta>0$. This equation may be viewed as a normal form of the supercritical bifurcation leading to translational motion. Such bifurcations are possible in simple physicochemical systems [3]. They are also known for localized spot patterns in reaction-diffusion models and correspond to the onset of their translational motion [10,12].

According to Eq. (1), the velocity $V$ is zero below the bifurcation point (i.e., for $\alpha<0$). Above this point, active motion with $V=\pm\sqrt{\alpha/\beta}$ is asymptotically established. The direction of this motion for an individual particle remains arbitrary and is determined by initial conditions. Rescaling time and introducing the new velocity variable $u=V\sqrt{\beta/\alpha}$, Eq. (1) can be written as

 

$$\dot{u} = u - u^3. \tag{2}$$

When a population of identical self-moving particles is considered, the velocity $u_i = \dot{x}_i$ of each particle $i$ will satisfy this dynamical equation.

Interactions between individuals may generally depend on both their relative positions and velocities. In this paper we assume that the interactions are pairwise and described by forces $f(x_i - x_j)$ that depend only on the difference of coordinates of two particles $i$ and $j$. We shall further assume that the interactions are attractive and depend linearly on the distance between the particles, i.e., $f(x_i - x_j) \propto (x_i - x_j)$. These attractive forces are supposed to model the interaction within the size ranges of the dynamical states considered below, where the population forms clouds of either oscillating or translational motion. The interaction could be extended to larger distances in order to represent, for instance, vanishing forces at infinity [22]. Additionally, the system may include noise that will be modeled by independent random forces $\xi_i(t)$ acting on individual particles. Noise prevents the collapse of the population, so that short-range repulsion [15,22] can here be ignored.

Under these conditions, the dynamical equations for a set of $N$ identical self-moving particles with coordinates $x_i(t)$ are

$$\ddot{x}_i + (\dot{x}_i^2 - 1)\dot{x}_i + \frac{a}{N}\sum_{j=1}^{N}(x_i - x_j) = \xi_i(t) \quad (i = 1, \ldots, N). \tag{3}$$

The coefficient $a$ characterizes the intensity of interactions and can be viewed as the parameter, specifying the strength of coupling in the population. Equations (3) constitute the basic model investigated in this paper. We shall assume that $\xi_i(t)$ are independent white noises of intensity $S$, so that $\langle \xi_i(t)\xi_j(t')\rangle = 2S\delta_{ij}\delta(t-t')$. Note that Eqs. (3) are invariant with respect to an arbitrary translation in the coordinate space.

Model (3) can behave as a system of globally coupled limit-cycle oscillators (cf. [23,24]). Introducing the average coordinate $\overline{x}(t)$ of the swarm,

$$\overline{x}(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(t), \tag{4}$$

Eqs. (3) in absence of noise read

$$\ddot{x}_i + (\dot{x}_i^2 - 1)\dot{x}_i + a(x_i - \overline{x}) = 0 \quad (i = 1, \ldots, N). \tag{5}$$

Thus, if the swarm does not move as a whole, i.e., $\overline{x}(t) = \text{constant}$, the particles perform persistent oscillations. In this state the phases of individual oscillations are random. Note that the spatial location $\overline{x}$ of an oscillating swarm is arbitrary.

In addition to the random oscillatory state, system (5) has two coherent collapsed states where the coordinates of all particles are identical, i.e., $x_i = \overline{x}$ for any $i$. These states correspond to uniform translational motion of the entire swarm with the velocity $u = \pm 1$. A simple analysis shows that the oscillatory state and both coherent traveling states are linearly stable for any positive parameter $a$. The final state of the population is determined by the initial conditions. Our numerical simulations show that, if the average velocity $\overline{u} = N^{-1}\Sigma_i u_i$ is initially close to zero, the oscillatory standing state is asymptotically reached. If, however, this initial average velocity is large enough, one of the two coherent traveling states will be approached.

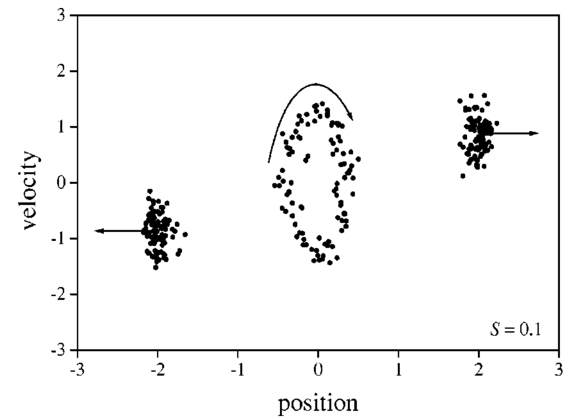


FIG. 1. Three snapshots of 100-particle systems with $a = 10$ and $S = 0.1$, in different dynamical regimes. The central ensemble corresponds to disordered oscillations along a noisy limit cycle. The other two ensembles stand for coherent clouds with opposite average velocities.

Since the particles either converge to coherent motion with constant velocity or to disordered oscillations with no average drift, the ensemble can be thought of as a *multistable system* with qualitatively different attractors. In the following, we focus our attention on how these attractors respond to the effect of noise. With this aim, we study Eq. (3) numerically. Integration is performed by means of a standard Euler scheme with a time step $\Delta t = 10^{-3}$ to $10^{-2}$. Most calculations correspond to ensembles of 100 particles, with the coupling intensity ranging from $a = 1$ to 100. Larger values of $a$ require smaller values of $\Delta t$. Noise is introduced by generating at each time step a random number $\xi$ with uniform distribution in the interval $(-\xi_0, \xi_0)$. This choice corresponds to having $S = \xi_0^2/6\Delta t$. In practice, $\xi_0$ is calculated for each given value of $S$. Initial conditions are selected at random, distributing the particles around $x = 0$ and $u = 0$ or 1 with a dispersion of the order of 0.5 in both variables. From each initial condition the system is left to evolve in the absence of noise until it reaches the state of disordered oscillations or coherent motion. Then, at $t = 30$, noise is switched on. Typical calculations extend up to $t \approx 1000$.

For small noise intensities $S \lesssim 0.1$, the stochastic perturbations to the trajectories preserve the characteristic features of the collective dynamics observed in the absence of noise. The completely collapsed state of the noiseless case transforms into a cloud of particles, which still moves coherently at a given velocity. Oscillatory orbits, meanwhile, proceed now along a noisy limit cycle. Figure 1 shows three snapshots of a system of 100 particles with $a = 10$, subjected to noise with $S = 0.1$. They started from different initial conditions, as described above. The arrows indicate the overall motion of each swarm.

Within coherent clouds, each particle performs an oscillatory noisy motion, which is superimposed to the collective

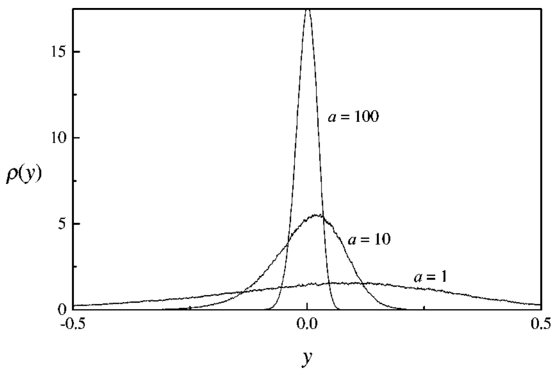


FIG. 2. Normalized profiles of coherent clouds as functions of the deviation from the average position, for different values of $a$ and $S=0.1$ in a 100-particle ensemble.

translation. The distribution of particles inside the clouds has a well-defined profile, shown in Fig. 2 for some values of $a$ in the case of positive velocity. The normalized distribution $\rho(y)$ is there plotted as a function of the coordinate relative to the average position $y_i = x_i - \bar{x}$. For decreasing $a$, the distribution becomes broader and more asymmetric, with an accumulation of particles at the front of the cloud.

The coherent traveling states of the population cease to exist at sufficiently high noise intensities and the swarm undergoes an abrupt transition to its random oscillatory state, characterized by the absence of the translational motion. This breakdown of coherent swarm motion is illustrated in Fig. 3. We see that if the noise is relatively weak [Fig. 3(a)], switching it on at $t=30$ only produces a slight decrease of the velocity of the coherent cloud, so that the average velocity $\bar{u}(t)$ exhibits fluctuations around a constant mean value $\bar{u} < 1$. If, however, the noise intensity exceeds a certain threshold, the effect of introducing noise is qualitatively different [Fig. 3(b)]. Within a certain time interval after the introduction of noise, the swarm continues to travel at a somewhat reduced, strongly fluctuating average velocity $\bar{u}(t)$. Then, it suddenly starts to decelerate and soon reaches a steady state where the average velocity $\bar{u}(t)$ fluctuates near zero. Inspection of the distribution of particles in the ensemble shows that in this state the system has been attracted to the noisy limit cycle mentioned above. We conclude that the system

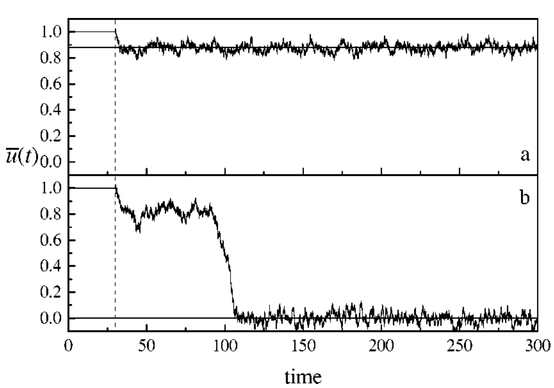


FIG. 3. Average velocity of 100-particle coherent clouds in two realizations at (a) $S=0.10$ and (b) 0.12, with $a=10$. Noise is switched on at $t=30$ (dashed line). The horizontal lines indicate the asymptotic mean values of $u(t)$.

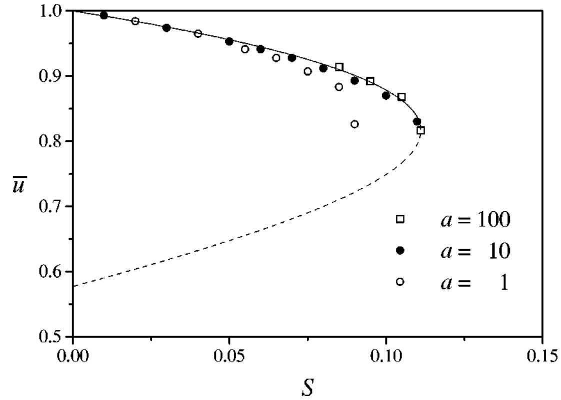


FIG. 4. The asymptotic mean velocity of 100-particle coherent clouds as a function of $S$, for different values of $a$. Symbols correspond to numerical measurements and lines stand for the analytical result, Eq. (13).

undergoes a *noise-induced transition* from a condition of multistability with two kinds of attractors to a situation where only one of them exists. The coherent clouds observed for small noise intensities are no longer possible for $S > S_c$, and the system is necessarily led to the state of noisy, disordered oscillations.

Figure 4 displays the dependence of the mean velocity $\bar{u}$ of the traveling swarm on the noise intensity $S$ for three different values of the coupling coefficient $a$. We see that the mean velocity monotonously decreases with the noise intensity, until a certain critical noise intensity is reached and the coherent swarm motion becomes impossible. The mean velocity at the critical point is still relatively large, $\bar{u} \approx 0.8$. The critical noise intensity $S_c$ becomes lower for smaller values of $a$. Note that the behavior of the swarm is characterized by a strong hysteresis. If the breakdown of the coherent motion has occurred, subsequently decreasing the noise intensity leaves the system in the oscillatory state with zero mean velocity, down to $S=0$.

An interesting property of the considered noise-induced transition is the divergence of the waiting time at the critical point. The waiting time $T_0$ is defined as the time at which the average velocity $\bar{u}(t)$ of the cloud first reaches zero (we measure this time starting from the moment $t=30$ when the noise is switched on). Figure 5 shows the waiting time $T_0$ as a function of $S$-$S_c$ in a log-log plot. We see that for very small values of $S$-$S_c$, this time decreases following a power law, $T_0 \propto (S\text{-}S_c)^{-\gamma}$, with $\gamma \approx 1.33$. Then, at about $S$-$S_c = 0.03$, the behavior changes to a power law with $\gamma \approx 0.52$. Straight dashed lines with slopes $-4/3$ and $-1/2$ have been plotted for reference.

The observed noise-induced transition between coherent clouds and disordered oscillations of the swarm can be explained by a simple approximate analytical approach. By summing all Eqs. (5) for different particles $i$ and taking into account that the noises acting on individual particles are not correlated, an evolution equation for the average swarm velocity $\bar{u}(t)$ is obtained:

$$\dot{\bar{u}} + \frac{1}{N}\sum_{i=1}^{N} \dot{x}_i^3 - \bar{u} = 0. \tag{6}$$

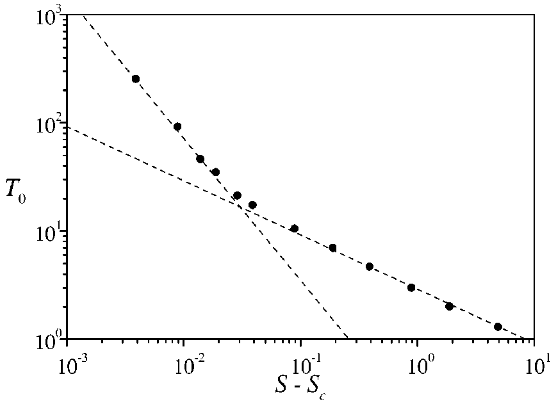


FIG. 5. Waiting time $T_0$, at which the average velocity of initially coherent clouds vanishes for the first time, as a function of $S$-$S_c$ for $a=10$ in a 100-particle system. The dashed lines have slopes $-1/2$ and $-4/3$.

Let us introduce for each particle its deviation $y_i=x_i-\overline{x}$ from the average position of the swarm. Then we can write

$$\frac{1}{N}\sum_{i=1}^{N}\dot{x}_i^3=\overline{u^3}+3\sigma\overline{u}+\frac{1}{N}\sum_{i=1}^{N}\dot{y}_i^3, \tag{7}$$

where $\sigma=N^{-1}\Sigma_i\dot{y}_i^2$ is the average square dispersion of the swarm. The last cubic term in this equation can be neglected if the distribution of particles in the traveling cloud is symmetric. As we have seen from numerical simulations (Fig. 2), this is indeed a good approximation for sufficiently large values of the coupling constant $a$. Within this approximation, Eq. (6) takes the form

$$\dot{\overline{u}}+(\overline{u^2}-1)\overline{u}+3\sigma\overline{u}=0. \tag{8}$$

On the other hand, deviations of particles from the center of the swarm obey the stochastic differential equation

$$\ddot{y}_i+(3\overline{u^2}-1)\dot{y}_i+ay_i+3\overline{u}(\dot{y}_i^2-\sigma)+\left(\dot{y}_i^3-\frac{1}{N}\sum_{i=1}^{N}\dot{y}_i^3\right)=\xi_i(t). \tag{9}$$

Assuming that the deviations of $\dot{y}_i$ are relatively small and linearizing this equation, we obtain

$$\ddot{y}_i+(3\overline{u^2}-1)\dot{y}_i+ay_i=\xi_i(t). \tag{10}$$

In this approximation the deviations for different particles $i$ represent statistically independent random processes. This allows us to replace the ensemble average in the dispersion $\sigma$ by the statistical average taken over independent random realizations of such processes, defined by Eq. (10).

Hence, we have derived a closed set of Eqs. (8) and (10) that approximately describe the swarm. We want to investigate steady statistical states of this system. The stationary solutions to Eq. (8) are $\overline{u}=\pm\sqrt{1-3\sigma}$ and $u=0$. The latter solution corresponds to the resting swarm.

Examining Eq. (10), we note that it describes damped oscillations only if $3\overline{u^2}-1>0$, i.e., only if the mean velocity of the swarm is sufficiently large. Under this condition, the stationary probability distribution for $y_i$ is readily found and the average square dispersion of velocities is obtained as

$$\sigma=\frac{S}{3\overline{u^2}-1}. \tag{11}$$

The algebraic equations for $\overline{u}$ and $\sigma$ can be solved, yielding the statistical dispersion of particles in the traveling swarm,

$$\sigma_{1,2}=\tfrac{1}{9}(1\pm\sqrt{1-9S}), \tag{12}$$

and its mean velocity

$$\overline{u^2_{1,2}}=\tfrac{1}{3}(2\pm\sqrt{1-9S}). \tag{13}$$

Thus, the traveling-state solutions disappear when the critical noise intensity $S_c=1/9=0.11\ldots$ is reached. At this critical point the mean swarm velocity is $\overline{u_c}=\sqrt{2/3}=0.82\ldots$ and the mean dispersion of particles in the cloud is $\sigma_c=1/9=0.11\ldots$

Below the breakdown threshold (for $S<S_c$), solution (13) has two branches shown by solid and dashed lines in Fig. 4. The lower branch is apparently unstable, since it approaches the value $\overline{u}=1/\sqrt{3}=0.58\ldots$ at $S=0$, i.e., in absence of the noise. A special property of the derived solution is that it does not depend on the parameter $a$.

Comparing the theoretical prediction with the numerically determined values of the mean swarm velocity, which are also plotted in Fig. 4, we can see that this approximation provides good estimates of the swarm velocity and the critical noise intensity when the parameter $a$ is relatively high ($a=100$ and $a=10$). At small values of $a$, the deviations from the numerical results become significant near the breakdown threshold. This can be understood if we take into account that, according to Fig. 2, the distribution of particles in a traveling swarm shows significant asymmetry for such a small value of $a$ and, therefore, our approximations are not valid.

For a standing swarm ($\overline{u}=0$), the deviations $y_i=x_i-\overline{x}$ obey in the limit $N\rightarrow\infty$ the nonlinear stochastic differential equation

$$\ddot{y}_i+(\dot{y}_i^2-1)\dot{y}_i+ay_i=\xi_i(t), \tag{14}$$

which is similar to the Van der Pol equation [25]. In this state, therefore, the particles in the swarm perform periodic limit-cycle oscillations with a random distribution of phases. This state exists for any noise intensity $S$ and is approached when the noise-induced breakdown of the coherent motion takes place at $S=S_c$.

Thus, we have found in this paper that a swarm of interacting, actively moving particles may show bistable behavior, i.e., they can be found either in a coherent state traveling at a fixed velocity, or in a rest state where the translational motion is absent and the individual particles perform oscillations around the center of the swarm. The bistability persists in the presence of noise if its intensity remains relatively low. Increasing the noise intensity leads to a sudden breakdown of the coherent traveling motion and a transition to the resting oscillatory state occurs. This behavior is different from the second-order phase transitions to coherent collective motion, which were found in the previously studied models [17,18]. We conjecture that the difference is related to the fact that in our model the interactions between self-propelled particles have a long range and extend over the entire swarm. It would be interesting to see how this behavior is modified when other interaction laws and systems with higher dimensionality are considered. Finally, we remark that, when formulated in terms of dynamical equations for individual interacting self-propelled particles, the problem shows significant similarities to synchronization and condensation in populations of globally coupled oscillators (see, e.g., [23,26]). The significant new aspect is that collapsed synchronous states correspond here to translational motion of the entire population.

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